51. $\sqrt{1+\log_{\sqrt{2}} x}=3$

$$c.a.:\begin{cases}x>0\\1+\log_{\sqrt{2}} x\geq 0\end{cases}\Rightarrow\begin{cases}x>0\\\log_{\sqrt{2}} x\geq -1\end{cases}\Rightarrow\begin{cases}x>0\\x\geq\dfrac{1}{\sqrt{2}}\end{cases}\Rightarrow x\geq\frac{1}{\sqrt{2}}$$

$$1+\log_{\sqrt{2}} x=9\Rightarrow\log_{\sqrt{2}} x=8\Rightarrow x=\sqrt{2}^{8}\Rightarrow \boxed{x=16}$$

52. $\log_{\frac{1}{3}}\log_{\frac{1}{3}}(5x+9)=0$

$$c.a.:\begin{cases}5x+9>0\\\log_{\frac{1}{3}}(5x+9)>0\end{cases}\Rightarrow\begin{cases}x>-\dfrac{9}{5}\\5x+9\leq 1\end{cases}\Rightarrow\begin{cases}x>-\dfrac{9}{5}\\x\leq-\dfrac{8}{5}\end{cases}\Rightarrow -\frac{9}{5}<x\leq-\frac{8}{5}$$

$$\log_{\frac{1}{3}}(5x+9)=1\Rightarrow 5x+9=\frac{1}{3}\Rightarrow 5x=-\frac{26}{3}\Rightarrow \boxed{x=-\frac{26}{15}}$$

53. $(x^2-1)\,2^{\log_2(x-2)}=\log_2 2^{x+1}$

$2^{\log_2(x-2)}=x-2$ infatti, per la definizione di logaritmo: $a^b=c\Leftrightarrow\log_a c=b$

cioè: $2^{\log_2(x-2)}=x-2\Leftrightarrow\log_2(x-2)=\log_2(x-2)$

$c.a.: x-2>0\Rightarrow x>2:\quad (x^2-1)(x-2)=x+1$

$(x-1)(x+1)(x-2)=x+1\Rightarrow x^2-3x+2=1$

$$x^2-3x+1=0\Rightarrow x_{1,2}=\frac{3\pm\sqrt{9-4}}{2}=\begin{cases}\dfrac{3+\sqrt{5}}{2}\\\dfrac{3-\sqrt{5}}{2}\ \textit{non acc.}\end{cases}$$

$$\boxed{x=\frac{3+\sqrt{5}}{2}}$$

54. $\ln^2 x+\ln x=0$

$c.a.: x>0\quad$ Pongo: $t=\ln x\Rightarrow t^2+t=0\Rightarrow t(t+1)=0$

$t=0\Rightarrow\ln x=0\Rightarrow \boxed{x=1}$

$t=-1\Rightarrow\ln x=-1\Rightarrow \boxed{x=\frac{1}{e}}$

55. $\ln^3 x + 2\ln^2 x - 3\ln x = 0$

$c.a.: x > 0$ Pongo: $t = \ln x \Rightarrow t^3 + 2t^2 - 3t = 0 \Rightarrow t(t^2 + 2t - 3) = 0$

$t_{1,2} = \dfrac{-1 \pm \sqrt{1+3}}{1} \left\langle \begin{matrix} 1 \\ -3 \end{matrix} \right.$

$t = 0 \Rightarrow \ln x = 0 \Rightarrow \boxed{x = 1}$

$t = 1 \Rightarrow \ln x = 1 \Rightarrow \boxed{x = e}$

$t = -3 \Rightarrow \ln x = -3 \Rightarrow \boxed{x = \dfrac{1}{e^3}}$

56. $|2\ln x + 1| = 4$

$c.a.: x > 0: \; 2\ln x + 1 = \pm 4$

$2\ln x = 3 \Rightarrow \ln x = \dfrac{3}{2} \Rightarrow x = e^{\frac{3}{2}} \Rightarrow \boxed{x = e\sqrt{e}}$

$2\ln x = -5 \Rightarrow \ln x = -\dfrac{5}{2} \Rightarrow x = e^{-\frac{5}{2}} \Rightarrow \boxed{x = \dfrac{1}{e^2\sqrt{e}}}$

57. $\ln x = \log x$

$c.a.: x > 0$ Applico la regola del cambiamento di base:

$\ln x = \dfrac{\ln x}{\ln 10} \Rightarrow (\ln 10 - 1)\ln x = 0 \Rightarrow \ln x = 0 \Rightarrow \boxed{x = 1}$

58. $\ln \dfrac{x^2 - 1}{x} = \ln 2$

$c.a.: \dfrac{x^2 - 1}{x} > 0 \Rightarrow \left| \begin{matrix} N > 0: x < -1 \vee x > 1 \\ D > 0; \; x > 0 \end{matrix} \right. \Rightarrow -1 < x < 0 \vee x > 1$

$\dfrac{x^2 - 1}{x} = 2 \Rightarrow \dfrac{x^2 - 1 - 2x}{x} = 0 \Rightarrow x^2 - 2x - 1 = 0 \Rightarrow$

$x_{1,2} = \dfrac{1 \pm \sqrt{1+1}}{1} \left\langle \begin{matrix} 1 + \sqrt{2} \\ 1 - \sqrt{2} \end{matrix} \right.$

$\boxed{x = 1 + \sqrt{2}; \quad x = 1 - \sqrt{2}}$

59. $\ln(e^x + e) = 2$

$c.a.: e^x + e > 0 \Rightarrow \forall x \in R$

$e^x + e = e^2 \Rightarrow e^x = e^2 - e \Rightarrow x = \ln e(e-1) \Rightarrow \boxed{x = 1 + \ln(e-1)}$

60. $\ln^2 x + 2|\ln x| - 15 = 0$

$c.a.: x > 0$ Pongo: $t = |\ln x| \Rightarrow t^2 + 2t - 15 = 0 \Rightarrow t_{1,2} = \dfrac{-1 \pm \sqrt{1+15}}{1}$

$t = 3 \Rightarrow |\ln x| = 3 \Rightarrow \ln x = \pm 3 \Rightarrow x = e^{\pm 3} \Rightarrow \boxed{x = e^3;\ x = \dfrac{1}{e^3}}$

$t = -5 \Rightarrow |\ln x| = -5 \Rightarrow imp.$

61. $\sqrt{\ln x} = 1 - \ln x$

$$c.a.: \begin{cases} x > 0 \\ \ln x \geq 0 \\ 1 - \ln x \geq 0 \end{cases} \Rightarrow \begin{cases} x > 0 \\ x \geq 1 \\ x \leq e \end{cases} \Rightarrow 1 \leq x \leq e$$

pongo: $\ln x = t$

$\sqrt{t} = 1 - t \Rightarrow t = 1 - 2t + t^2 \Rightarrow t^2 - 3t + 1 = 0$

$$t_{1,2} = \frac{3 \pm \sqrt{9-4}}{2} \begin{cases} \dfrac{3+\sqrt{5}}{2} \Rightarrow \ln x = \dfrac{3+\sqrt{5}}{2} \quad non\ acc. \\ \dfrac{3-\sqrt{5}}{2} \Rightarrow \ln x = \dfrac{3-\sqrt{5}}{2} \Rightarrow \end{cases}$$

$\boxed{x = e^{\frac{3-\sqrt{5}}{2}}}$